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Tutorial : 2

1) Prove that $\log(1+e^{2i\theta}) = \log(2\cos\theta)+i\theta$

→ Given:- $\log(1+e^{2i\theta}) = \log(2\cos\theta)+i\theta$

$$\log(1+e^{2i\theta}) = \log(1+\cos 2\theta + i\sin 2\theta)$$
$$= \log(1+2\cos^2\theta - 1 + 2i\sin\theta)$$
$$= \log(2\cos^2\theta + 2i\sin\theta\cos\theta)$$
$$= \log[2\cos\theta(\cos\theta + i\sin\theta)]$$
$$= \log[2\cos\theta\, e^{i\theta}]$$
$$= \log[2\cos\theta) + \log^{i\theta}$$
$$\log(1+e^{2i\theta}) = \log(2\cos\theta)+i\theta.$$

2) If $\tan(A+iB) = (x+iy)$, prove that

i) $\tan 2A = \dfrac{2x}{1-x^2-y^2}$ ii) $\tanh 2B = \dfrac{2y}{1+x^2+y^2}$

→ Given :- $\tan(A+iB) = x+iy$ —— ①

&

$\tan(A-iB) = x-iy$ —— ②

Let ~~Add eq^n ① & ②~~

$(A+iB) = C$ —— ③

$(A-iB) = D$ —— ④

Adding eq. ③ & ④

$\therefore 2A = C+D$

Apply tan on both sides

$$\tan 2A = \tan(C+D)$$

$$\tan A+B = \frac{\tan A + \tan B}{1-\tan A \tan B}$$

$$= \frac{\tan C + \tan D}{1-\tan C \tan D}$$

$$= \frac{\tan A+iB + \tan A-iB}{1-\tan A+iB \cdot \tan A-iB}$$

$$= \frac{x+iy + x-iy}{1-(x+iy)(x-iy)}$$

$$\boxed{\therefore \tan 2A = \frac{2x}{1-x^2-y^2}}$$

$\because (A+B)(A-B) = A^2-B^2$

Substracting eq^n 3 & 4 we get

$$2iB = C-D$$

Apply tan on both side

$$\tan(2iB) = \tan(C-D)$$

$$i\tan h2B = \frac{\tan C - \tan D}{1-\tan C \tan D}$$

$$= \frac{x+iy-(x-iy)}{1+(x+iy)(x-iy)}$$

$$= \frac{2iy}{1+x^2+y^2}$$

$$\boxed{\therefore \tan h2B = \frac{2y}{1+x^2+y^2}}$$

$z = re^{i\theta}$

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3) If $\cos(\theta + i\phi) = Re^{i\alpha}$, show that

$$\phi = \frac{1}{2}\log_e\left(\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right)$$

→ Given :- $\cos(\theta + i\phi) = Re^{i\alpha}$

we know that,

$A = \theta$
$\phi = i\phi$

$r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore Re^{i\alpha} = R(\cos\alpha + i\sin\alpha)$

∴ By using trigonometric formula

— $[\cos(A+B) = \cos A \cdot \cos B - \sin A . \sin B]$

$\cos\theta . \cos(i\phi) - \sin\theta . \sin(i\phi) = R(\cos\alpha + i\sin\alpha)$

$\cos\theta . \cosh\phi - i\sin\theta . \sinh\phi = R\cos\alpha + iR\sin\alpha$

$(\because \sin(ix) = i\sinh x$ and $\cos(ix) = \cosh x)$

Equating real and imaginary parts on both sides

$\cos\theta . \cosh\phi = R\cos\alpha$ and

$-\sin\theta \sin\phi = R\sin\alpha$ — ②

$$\left[\cosh\phi = \frac{R\cos\alpha}{\cos\theta} \text{ and } \sin\phi = \frac{R\sin\alpha}{-\sin\theta}\right]$$

$$\frac{\sinh\phi}{\cosh\phi} = \frac{\frac{R\sin\alpha}{-\sin\theta}}{\frac{R\cos\alpha}{\cos\theta}}$$

$$\tanh\phi = \frac{-\sin\alpha\cos\theta}{\cos\alpha . \sin\theta}$$

$$\phi = \tanh^{-1}\left[\frac{-\sin\alpha . \cos\theta}{\cos\alpha . \sin\theta}\right]$$

Using inverse hyperbolic formula

$$-\left[\tanh^{-1} x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right]$$

$$= \frac{1}{2}\log\left[\frac{1-\left(\frac{\sin\alpha.\cos\theta}{\cos\alpha.\sin\theta}\right)}{1+\left(\frac{\sin\alpha\cos\theta}{\cos\alpha.\sin\theta}\right)}\right]$$

$$= \frac{1}{2}\log\left[\frac{\cos\alpha.\sin\theta - \sin\alpha.\cos\theta}{\cos\alpha.\sin\theta + \sin\alpha.\cos\theta}\right]$$

Using standard trigonometric formula

$$-\left[\begin{array}{l}\sin(A+B) = \sin A.\cos B + \cos A.\sin B \\ \sin(A-B) = \sin A.\cos B - \cos A \sin B\end{array}\right]$$

$$\phi = \frac{1}{2}\log\left[\frac{\sin(\theta-\alpha)}{\sin(\theta+\alpha)}\right]$$

$$\left(\frac{\cos\alpha}{\cos\theta}\right)^2 - \left(\frac{-\sin\alpha}{\sin\theta}\right)^2 = 1$$

$$\therefore \sin^2\theta.\cos^2\alpha - \cos^2\theta.\sin^2\alpha = \cos^2\theta.\sin^2\theta$$

$$\sin^2\theta[1-\sin^2\alpha] - [1-\sin^2\theta]\sin^2\alpha = (1-\sin^2\theta)\sin^2\theta$$

$$\sin^2\theta - \sin^2\theta.\sin^2\alpha - \sin^2\alpha + \sin^2\theta.\sin^2\alpha = \sin^2\theta - \sin^4\theta$$

$$-\sin^2\alpha = -\sin^4\theta$$

$$\sin^2\alpha = \sin^4\theta$$

$$\sin^4\theta = \sin^2\alpha$$

Taking square root of both sides

$$\boxed{\therefore \sin^2\theta = \pm\sin\alpha}$$ Hence, proved.

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4) For $x = \sqrt{3}$, find the value of $\tanh(\log x)$

→ Given :- To find the $\tanh(\log x)$ if $x = \sqrt{3}$

we know that $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

Put $x = \log\sqrt{3}$

$$\tan\log\sqrt{3} = \frac{e^{\log\sqrt{3}} - e^{-\log\sqrt{3}}}{e^{\log\sqrt{3}} + e^{-\log\sqrt{3}}}$$

$$= \frac{e^{\log\sqrt{3}} - e^{\frac{1}{\log\sqrt{3}}}}{e^{\log\sqrt{3}} + \frac{1}{e^{\log\sqrt{3}}}}$$

$$= \frac{\sqrt{3} - \frac{1}{\sqrt{3}}}{\sqrt{3} + \frac{1}{\sqrt{3}}} = \frac{\frac{3-1}{\cancel{\sqrt{3}}}}{\frac{3+1}{\cancel{\sqrt{3}}}}$$

$$= \frac{3-1}{3+1} = \frac{2}{4}$$

$$= \frac{1}{2}$$

for $x = \sqrt{3} \rightarrow \tanh(\log x) = \frac{1}{2}$

5) If $y = \log\tan x$, prove that

i) $\sinh y = \frac{1}{2}(\tan^n x - \cot^n x)$

ii) $2\cosh ny \operatorname{cosec} 2x = \cosh(n+1)y + \cosh(n-1)y$

→ Given :- $y = \log(\tan x)$

$\therefore \tan x = e^{y}$ and $\cot x = e^{-y}$

$\tan^n x = e^{ny}$ and $\cot^n x = e^{-ny}$

$\therefore \frac{1}{2}(\tan^n x - \cot^n x) = \frac{1}{2}(e^{ny} - e^{-ny})$

$= \sinh ny$

ii) $2\cosh ny \operatorname{cosec} 2x = \cosh(n+1)y + \cosh(n-1)y$

$RHS = \cosh(n+1)y + \cosh(n-1)y$ — ①

$= 2\cosh ny \cosh y$

$\cosh y = \frac{e^{y} + e^{-y}}{2}$

$= \frac{\tan x + \cot x}{2}$

$= \frac{\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}}{2}$

$= \frac{\sin^2 x + \cos^2 x}{2 \sin x \cdot \cos x}$

$= \frac{1}{\sin 2x}$

$= \operatorname{cosec} 2x$

Putting in ① we got

$RHS = 2\cosh ny \cdot \operatorname{cosec} 2x$

$= LHS$

6) Prove that $\log(e^{i\alpha}+e^{i\beta}) = \log\left(2\cos\frac{\alpha-\beta}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right)$

→ ~~Given :-~~ Sol^n :-

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log[(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)]$$

① Using Euler's Identity ; .... $[e^{i\theta} = \cos\theta + i\sin\theta]$
or $e^{ix} = \cos x + i\sin x$

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log[(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)]$$

Using standard trigonometric formulae

$$\ldots \left[\begin{array}{l} \cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \text{ and} \\ \sin A + \sin A = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{array} \right]$$

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log\left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + i2\sin\left[\frac{\alpha+\beta}{2}\right]\cos\left[\frac{\alpha-\beta}{2}\right]\right\}$$

Common out $2\cos\left(\frac{\alpha-\beta}{2}\right)$ from RHS bracket

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]\right.$$

— $[e^{ix} = \cos x + i\sin x]$

Using Euler's Identity :

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right) e^{i\left(\frac{\alpha+\beta}{2}\right)}\right\}$$

Using property of logarithm

$$-[\log(AB) = \log A + \log B]$$

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log\left[2\cos\left(\frac{\alpha-\beta}{2}\right)\right] + \log\left\{e^{i\left(\frac{\alpha+\beta}{2}\right)}\right\}$$

$$\log\{e^{i\alpha}+e^{i\beta}\} = \log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right)\right\} + i\left(\frac{\alpha+\beta}{2}\right)$$

Hence proved.

2) Show that for any real number a and b,

$$e^{2ai\cot^{-1}b} \times \left\{\frac{bi-1}{bi+1}\right\}^{-a} = 1$$

⇒ Sol^n :- we have $\frac{bi-1}{bi+1}$

we can also write it has

$$= \frac{i(b+i)}{i(b-i)}$$

$$= \frac{b+i}{b-i}$$

$$\therefore \left\{\frac{bi-1}{bi+1}\right\}^{-a} = \left\{\frac{b+i}{b-i}\right\}^{-a} = z \text{ (say)}.$$

or

Consider $\frac{bi-1}{bi+1} = \frac{\left[bi - i \times \frac{1}{i}\right]}{\left[bi + i \times \frac{1}{i}\right]}$

$$= \frac{i\left(b - \frac{1}{i}\right)}{i\left(b + \frac{1}{i}\right)} = \frac{b+i}{b-i} \quad \text{--- (1)}$$

$$\text{---} \left[\because \frac{1}{i} = i\right]$$

Now consider, $b+i = r\left[\cos\theta + i\sin\theta\right] = re^{i\theta}$ --- (2)

We know, If $z = x+iy = r\left[\cos\theta + i\sin\theta\right]$ is a complex no. then modulus of the complex no. z is $r = |z| = \sqrt{x^2+y^2}$

& $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$r = \sqrt{b^2+1} \qquad \& \qquad \theta = \tan^{-1}\left(\frac{1}{b}\right)$$

we know, if

$$z = x+iy = r\left[\cos\theta + i\sin\theta\right] \text{ then } x-iy = r\left[\cos\theta - i\sin\theta\right]$$

$$b-i = r\left[\cos\theta - i\sin\theta\right] = re^{-i\theta} \quad \text{--- (3)}$$

Substitute the values from Eq^n (2) & (3) in Eq^n (1) we get,

$$\therefore \frac{bi-1}{bi+1} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{i\theta+i\theta} = e^{2i\theta}$$

$$\left(\frac{bi-1}{bi+1}\right)^{-a} = \left(e^{2i\theta}\right)^{-a} = e^{-2ai\theta}$$

$\sin\theta = \tan^{-1}\left(\frac{1}{b}\right) = \cot^{-1} b$

$\left(\frac{bi-1}{bi+1}\right)^{-a} = e^{-2ia\tan^{-1}\left(\frac{1}{b}\right)} = e^{-2ia\cot^{-1} b}$

$\left[\frac{bi-1}{bi+1}\right]^{-a} = e^{-2ai\cot^{-1} b}$

$\therefore e^{2ia\cot^{-1} b} \cdot \left[\frac{bi-1}{bi+1}\right]^{a} = 1$

8) Solve for z, if $e^z = 1+i\sqrt{3}$

$\rightarrow$ $e^z = 1+\sqrt{3}i$

Above can be represented in polar form as

$$1+\sqrt{3}i = r(\cos\theta + i\sin\theta)$$

Here,

$r = |1+\sqrt{3}i|$

$r\cos\theta = 1$

$r\sin\theta = \sqrt{3}$

we know,

$r = \sqrt{(1)^2 + (\sqrt{3})^2}$

$= \sqrt{1+3}$

$|1+\sqrt{3}i| = 2$

$r\cos\theta = 2\cos\theta = 1$

$\cos\theta = \frac{1}{2}$

And $r\sin\theta = 2\sin\theta = \sqrt{3}$

$$\sin\theta = \frac{\sqrt{3}}{2}$$

we know,

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\& \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

So, in polar form

$$r = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

We know $\cos x$ & $\sin x$ are cyclic function with period 2π, So,

$$r = 2\left[\cos\left(2n\pi + \frac{\pi}{3}\right) + i\sin\left(2n\pi + \frac{\pi}{3}\right)\right]$$

$$r = 2\left[\cos\frac{(6n+1)\pi}{3} + i\sin\frac{(6n+1)\pi}{3}\right]$$

Where n is an integer,

We know Euler's form of

$$\cos\theta + i\sin\theta = e^{i\theta}$$

$$\& \quad 2 = e^{\log 2}$$

$$r = 2\left[\cos\frac{(6n+1)\pi}{3} + i\sin\frac{(6n+1)\pi}{3}\right]$$

$$= e^{\log 2}\, e^{i\frac{(6n+1)\pi}{3}}$$

$$= e^{\left(\log 2 + i\left(\frac{(6n+1)\pi}{3}\right)\right)}$$

Thus $\boxed{z = \log 2 + i\left(\frac{(6n+1)\pi}{3}\right)}$

Where n is integer

c) Separate into real & imaginary parts the expression i^{i}.

→ Sol^n:- $i^{i} = e^{\log\{i^{i}\}}$

$= e^{i \log i}$ —— ①

$i^{i} = e^{\log i^{i}}$

$= e^{i \log i}$

$= e^{i\{\log 1 + i(2n\pi + \frac{\pi}{2})\}}$

$= e^{i\{i(2n\pi + \frac{\pi}{2})\}}$

$= e^{-(2n\pi + \frac{\pi}{2})}$

Let $\log i = \left\{\log 1 + i\left(2m\pi + \frac{\pi}{2}\right)\right\}$

$= i\left(2m\pi + \frac{\pi}{2}\right)$

∴ Substituting for i^{i} and $\log i$ in ①, we get

$i^{i} = e^{e^{-(2n\pi + \frac{\pi}{2})} \cdot i\left(2m\pi + \frac{\pi}{2}\right)}$

$= e^{i\theta}$

$= \cos\theta + i\sin\theta$

where $\theta = \left[\left(2m\pi + \frac{\pi}{2}\right) e^{-(2n\pi + \frac{\pi}{2})}\right]$

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10) Find general & principal value of the logarithm of $\log_2(-3)$.

→ Sol^n :- $\log_2(3) = \frac{\log_e(-3)}{\log_e(2)}$ — changing the base of logarithm to e

Now $-3 = 3(\cos\pi + i\sin\pi)$ $\therefore r = 3$ & $\theta = \pi$

$2 = 2(\cos 0 + i\sin 0)$ $r = 3$ & $\theta = 0$

$$\frac{\log(-3)}{\log(2)} = \frac{\log 3 + i(2n\pi + \pi)}{\log 2 + i(2m\pi + 0)}$$

$$= \frac{\log 3 + i(2n+1)\pi}{\log 2 + i(2m\pi)} \times \left\{\frac{\log 2 - 2m\pi i}{\log 2 - 2m\pi i}\right\}$$

$$= \frac{\{\log 3 \times \log 2 + 2m(2n+1)\pi\} + i\{(2n+1)\pi\log 2 - 2m\pi\log 3\}}{(\log 2)^2 + 4m^2\pi^2}$$

Principal value is obtained by putting $m = 0$, $n = 0$.

$$\frac{\log(-3)}{\log 2} = \frac{\log 3 \times \log 2 + \pi i \log 2}{(\log 2)^2}$$

$$= \frac{\log 3 + \pi i}{(\log 2)}.$$