

# BATU-EXAM

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at MET Bhujbal Knowledge City

Engg Maths 2 Department

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## Tutorial : 2

1) Prove that  $\log(1 + e^{2i\theta}) = \log(2\cos\theta) + i\theta$

→ Given:  $\log(1 + e^{2i\theta}) = \log(2\cos\theta) + i\theta$

$$\begin{aligned}\log(1 + e^{2i\theta}) &= \log(1 + \cos 2\theta + i\sin 2\theta) \\ &= \log(1 + 2\cos^2\theta - 1 + 2i\sin\theta\cos\theta) \\ &= \log(2\cos^2\theta + 2i\sin\theta\cos\theta)\end{aligned}$$

$$= \log[2\cos\theta(\cos\theta + i\sin\theta)]$$

$$= \log [2\cos\theta e^{i\theta}]$$

$$= \log(2\cos\theta) + \log e^{i\theta}$$

$$\log(1 + e^{2i\theta}) = \log(2\cos\theta) + i\theta.$$

2) If  $\tan(A + iB) = (x + iy)$ , prove that

i)  $\tan 2A = \frac{2x}{1 - x^2 - y^2}$

ii)  $\tan 2B = \frac{2y}{1 + x^2 + y^2}$

→ Given:  $\tan(A + iB) = x + iy$  — (1)

&  
 $\tan(A - iB) = x - iy$  — (2)

Let Add eq<sup>n</sup> (1) & (2)

$$\tan(A + iB) = C \text{ — (3)}$$

$$\tan(A - iB) = D \text{ — (4)}$$

Adding eq. (3) & (4)

$$2A = C + D$$

Apply tan on both sides

$$\tan 2A = \tan (C+D)$$

$$= \frac{\tan C + \tan D}{1 - \tan C \tan D}$$

$$= \frac{\tan A + iB + \tan A - iB}{1 - \tan A + iB \cdot \tan A - iB}$$

$$= \frac{x+iy + x-iy}{1 - (x+iy)(x-iy)}$$

$$\therefore \tan 2A = \frac{2x}{1-x^2-y^2}$$

$$\tan A+B = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\therefore (A+B)(A-B) = A^2 - B^2$$

Subtracting eq<sup>n</sup> (3) & (4) we get

$$2iB = C - D$$

Apply tan on both side

$$\tan (2iB) = \tan (C - D)$$

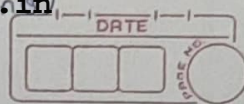
$$i \tan 2B = \frac{\tan C - \tan D}{1 - \tan C \tan D}$$

$$= \frac{x+iy - (x-iy)}{1 + (x+iy)(x-iy)}$$

$$= \frac{2iy}{1+x^2+y^2}$$

$$\therefore \tan 2B = \frac{2y}{1+x^2+y^2}$$

$$z = r e^{i\theta}$$



3) If  $\cos(\theta + i\phi) = R e^{i\alpha}$ , show that  

$$\phi = \frac{1}{2} \log \left( \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} \right)$$

→ Given:  $\cos(\theta + i\phi) = R e^{i\alpha}$

we know that

$$A = \theta \\ \phi = i\phi$$

$$r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

$$\therefore R e^{i\alpha} = R(\cos\alpha + i\sin\alpha)$$

∴ By using trigonometric formula

$$\text{--- } (\cos(A+B) = \cos A \cos B - \sin A \sin B)$$

$$\cos\theta \cdot \cos(i\phi) - \sin\theta \cdot \sin(i\phi) = R(\cos\alpha + i\sin\alpha)$$

$$\cos\theta \cdot \cosh\phi - i\sin\theta \cdot \sinh\phi = R\cos\alpha + iR\sin\alpha$$

$$(\because \sin(ix) = i\sin hx \text{ and } \cos(ix) = \cosh x)$$

Equating real and imaginary parts on both sides

$$\cos\theta \cdot \cosh\phi = R\cos\alpha \text{ and}$$

$$-i\sin\theta \sinh\phi = iR\sin\alpha \text{ --- (2)}$$

$$\left[ \cosh\phi = \frac{R\cos\alpha}{\cos\theta} \text{ and } \sinh\phi = \frac{R\sin\alpha}{-\sin\theta} \right]$$

$$\frac{\sinh\phi}{\cosh\phi} = \frac{R\sin\alpha}{\frac{R\cos\alpha}{\cos\theta}}$$

$$\tanh\phi = \frac{-\sin\alpha \cos\theta}{\cos\alpha \cdot \sin\theta}$$

$$\phi = \tanh^{-1} \left[ \frac{-\sin\alpha \cdot \cos\theta}{\cos\alpha \cdot \sin\theta} \right]$$

Using inverse hyperbolic formula

$$\rightarrow \left[ \tanh^{-1} x = \frac{1}{2} \log \left( \frac{1+x}{1-x} \right) \right]$$

$$= \frac{1}{2} \log \left[ \frac{1 - \left( \frac{\sin \alpha \cdot \cos \theta}{\cos \alpha \cdot \sin \theta} \right)}{1 + \left( \frac{\sin \alpha \cdot \cos \theta}{\cos \alpha \cdot \sin \theta} \right)} \right]$$

$$= \frac{1}{2} \log \left[ \frac{\cos \alpha \cdot \sin \theta - \sin \alpha \cdot \cos \theta}{\cos \alpha \cdot \sin \theta + \sin \alpha \cdot \cos \theta} \right]$$

Using standard trigonometric formula

$$\rightarrow \left[ \begin{array}{l} \sin(A+B) = \sin A \cdot \cos B + \cos A \cdot \sin B \\ \sin(A-B) = \sin A \cdot \cos B - \cos A \cdot \sin B \end{array} \right]$$

$$\phi = \frac{1}{2} \log \left[ \frac{\sin(\theta - \alpha)}{\sin(\theta + \alpha)} \right]$$

$$\left( \frac{\cos \alpha}{\cos \theta} \right)^2 - \left( \frac{-\sin \alpha}{\sin \theta} \right)^2 = 1$$

$$\begin{aligned} \therefore \sin^2 \theta \cdot \cos^2 \alpha - \cos^2 \theta \cdot \sin^2 \alpha &= \cos^2 \theta \cdot \sin^2 \theta \\ \sin^2 \theta [1 - \sin^2 \alpha] - [1 - \sin^2 \theta] \sin^2 \alpha &= (1 - \sin^2 \theta) \sin^2 \theta \\ \sin^2 \theta - \sin^2 \theta \cdot \sin^2 \alpha - \sin^2 \alpha + \sin^2 \theta \cdot \sin^2 \alpha &= \sin^2 \theta - \sin^2 \alpha \end{aligned}$$

$$-\sin^2 \alpha = -\sin^2 \theta$$

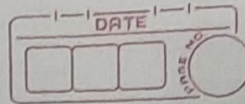
$$\sin^2 \alpha = \sin^2 \theta$$

$$\sin^2 \theta = \sin^2 \alpha$$

Taking square root of both sides

$$\boxed{\therefore \sin^2 \theta = \pm \sin^2 \alpha}$$

Hence, proved.



4) For  $x = \sqrt{3}$ , find the value of  $\tanh(\log x)$   
 → Given:- To find the  $\tanh(\log x)$  if  $x = \sqrt{3}$

we know that  $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Put  $x = \log \sqrt{3}$

$$\tanh \log \sqrt{3} = \frac{e^{\log \sqrt{3}} - e^{-\log \sqrt{3}}}{e^{\log \sqrt{3}} + e^{-\log \sqrt{3}}}$$

$$= \frac{e^{\log \sqrt{3}} - e^{\frac{1}{\log \sqrt{3}}}}{e^{\log \sqrt{3}} + \frac{1}{e^{\log \sqrt{3}}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + \frac{1}{\sqrt{3}}}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3} + 1} = \frac{3 - 1}{\sqrt{3} + 1}$$

$$= \frac{2}{\sqrt{3} + 1}$$

$$= \frac{2}{3 + 1} = \frac{2}{4}$$

$$= \frac{1}{2}$$

for  $x = \sqrt{3} \rightarrow \tanh(\log x) = \frac{1}{2}$

5) If  $y = \log \tan x$ , prove that

i)  $\sinh y = \frac{1}{2} (\tan^n x - \cot^n x)$

ii)  $2 \cosh ny \operatorname{cosec} 2x = \cosh(n+1)y + \cosh(n-1)y$

→ Given:-  $y = \log(\tan x)$

$$\therefore \tan x = e^y \quad \text{and} \quad \cot x = e^{-y}$$

$$\tan^n x = e^{ny} \quad \text{and} \quad \cot^n x = e^{-ny}$$

$$\therefore \frac{1}{2} (\tan^n x - \cot^n x) = \frac{1}{2} (e^{ny} - e^{-ny})$$

$$= \sinh ny$$

ii)  $2 \cosh ny \operatorname{cosec} 2x = \cosh (n+1)y + \cosh (n-1)y$

$$\text{RHS} = \cosh (n+1)y + \cosh (n-1)y \quad \text{--- (1)}$$

$$= 2 \cosh ny \cosh y$$

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

$$= \frac{\tan x + \cot x}{2}$$

$$= \frac{\sin x + \cos x}{\cos x \sin x}$$

$$= \frac{\sin^2 x + \cos^2 x}{2 \sin x \cos x}$$

$$= \frac{1}{\sin 2x}$$

$$= \operatorname{cosec} 2x$$

Putting in (1) we get

$$\text{RHS} = 2 \cosh ny \operatorname{cosec} 2x$$

$$= \text{LHS}$$

6) Prove that  $\log(e^{i\alpha} + e^{i\beta}) = \log\left(\frac{2\cos\frac{\alpha-\beta}{2}}{2}\right) + i\left(\frac{\alpha+\beta}{2}\right)$

→ Given + Sol<sup>n</sup> :-

$$\log\{e^{i\alpha} + e^{i\beta}\} = \log[(\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta)]$$

Using Euler's Identity : ...  $[e^{i\theta} = \cos\theta + i\sin\theta]$

$$\log\{e^{i\alpha} + e^{i\beta}\} = \log[(\cos\alpha + \cos\beta) + i(\sin\alpha + \sin\beta)]$$

Using standard trigonometric formulae

$$\left[ \begin{aligned} \cos A + \cos B &= 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \text{ and} \\ \sin A + \sin B &= 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right) \end{aligned} \right]$$

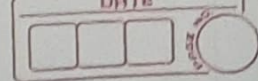
$$\log\{e^{i\alpha} + e^{i\beta}\} = \log\left\{2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right) + i2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)\right\}$$

Common out  $2\cos\left(\frac{\alpha-\beta}{2}\right)$  from RHS bracket

$$\log\{e^{i\alpha} + e^{i\beta}\} = \log\left\{2\cos\left(\frac{\alpha-\beta}{2}\right)\left[\cos\left(\frac{\alpha+\beta}{2}\right) + i\sin\left(\frac{\alpha+\beta}{2}\right)\right]\right\}$$

$$\leftarrow [e^{ix} = \cos x + i\sin x]$$





Using Euler's Identity:

$$\log \{e^{i\alpha} + e^{i\beta}\} = \log \left\{ 2 \cos \left( \frac{\alpha - \beta}{2} \right) \cdot e^{i \left( \frac{\alpha + \beta}{2} \right)} \right\}$$

Using property of logarithm

$$[\log (AB) = \log A + \log B]$$

$$\log \{e^{i\alpha} + e^{i\beta}\} = \log \left[ 2 \cos \left( \frac{\alpha - \beta}{2} \right) \right] + \log \left\{ e^{i \left( \frac{\alpha + \beta}{2} \right)} \right\}$$

$$\log \{e^{i\alpha} + e^{i\beta}\} = \log \left\{ 2 \cos \left( \frac{\alpha - \beta}{2} \right) \right\} + i \left( \frac{\alpha + \beta}{2} \right)$$

hence proved.

2) Show that for any real number  $a$  and  $b$ ,

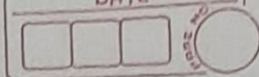
$$e^{2aica^{-1}b} \times \left\{ \frac{bi-1}{bi+1} \right\}^{-a} = 1$$

⇒ Sol<sup>n</sup> :- we have  $\frac{bi-1}{bi+1}$

we can also write it has

$$\begin{aligned} &= \frac{i(b+i)}{i(b-i)} \\ &= \frac{b+i}{b-i} \end{aligned}$$

$$\therefore \left\{ \frac{bi-1}{bi+1} \right\}^{-a} = \left\{ \frac{b+i}{b-i} \right\}^{-a} = z \text{ (say).}$$



or

$$\text{Consider } \frac{bi-1}{bi+1} = \frac{\left[bi - i \times \frac{1}{i}\right]}{\left[bi + i \times \frac{1}{i}\right]}$$

$$= \frac{i \left(b - \frac{1}{i}\right)}{i \left(b + \frac{1}{i}\right)} = \frac{b+i}{b-i} \quad \text{--- (1)}$$

Now consider  $b+i = r[\cos\theta + i\sin\theta] = re^{i\theta}$  --- (2)

We know, if  $z = x+iy = r[\cos\theta + i\sin\theta]$  is a complex no. then modulus of the complex no.  $z$  is  $r = |z| = \sqrt{x^2+y^2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$$r = \sqrt{b^2+1} \quad \& \quad \theta = \tan^{-1}\left(\frac{1}{b}\right)$$

we know, if

$$z = x+iy = r[\cos\theta + i\sin\theta] \text{ then } \overline{z} = r[\cos\theta - i\sin\theta]$$

$$b-i = r[\cos\theta - i\sin\theta] = r\overline{e^{i\theta}} \quad \text{--- (3)}$$

Substitute the values from Eq<sup>n</sup> (2) & (3) in Eq<sup>n</sup> (1) we get,

$$\therefore \frac{bi-1}{bi+1} = \frac{re^{i\theta}}{re^{-i\theta}} = e^{i\theta+i\theta} = e^{2i\theta}$$

$$\left(\frac{bi-1}{bi+1}\right)^{-a} = (e^{2i\theta})^{-a} = e^{-2ai\theta}$$

$$\sin \theta = \tan^{-1} \left( \frac{1}{b} \right) = \cot^{-1} b$$

$$\left( \frac{bi-1}{bi+1} \right)^{-a} = e^{-2ia \tan^{-1} \left( \frac{1}{b} \right)} = e^{-2ia \cot^{-1} b}$$

$$\left[ \frac{bi-1}{bi+1} \right]^{-a} = e^{-2ia \cot^{-1} b}$$

$$\therefore e^{2ia \cot^{-1} b} \cdot \left[ \frac{bi-1}{bi+1} \right]^a = 1$$

8) Solve for  $z$ , if  $e^z = 1 + i\sqrt{3}$

$$\rightarrow e^z = 1 + i\sqrt{3}$$

Above can be represented in polar form as

$$1 + i\sqrt{3} = r(\cos \theta + i \sin \theta)$$

Here,

$$r = |1 + i\sqrt{3}|$$

$$r \cos \theta = 1$$

$$r \sin \theta = \sqrt{3}$$

We know,

$$r = \sqrt{1^2 + (\sqrt{3})^2}$$

$$= \sqrt{1+3}$$

$$|1 + i\sqrt{3}| = 2$$

$$r \cos \theta = 2 \cos \theta = 1$$

$$\cos \theta = \frac{1}{2}$$

$$2$$

$$\text{And } r \sin \theta = 2 \sin \theta = \sqrt{3}$$

$$\text{we get } \sin \theta = \frac{\sqrt{3}}{2}$$

we know,

$$\cos \frac{\pi}{3} = \frac{1}{2}$$

$$\& \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

So, in polar form

$$r = 2 \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

We know  $\cos x$  &  $\sin x$  are cyclic functions with period  $2\pi$ , So,

$$r = 2 \left[ \cos \left( 2n\pi + \frac{\pi}{3} \right) + i \sin \left( 2n\pi + \frac{\pi}{3} \right) \right]$$

$$r = 2 \left[ \cos \frac{(6n+1)\pi}{3} + i \sin \frac{(6n+1)\pi}{3} \right]$$

Where  $n$  is an integer,

We know Euler's form of

$$\cos \theta + i \sin \theta = e^{i\theta}$$

$$\& 2 = e^{\log 2}$$

$$r = 2 \left[ \cos \frac{(6n+1)\pi}{3} + i \sin \frac{(6n+1)\pi}{3} \right]$$

$$= e^{\log 2} e^{i \frac{(6n+1)\pi}{3}}$$

$$= e^{(\log 2 + i \frac{(6n+1)\pi}{3})}$$

Thus 
$$z = \log z + i \left( \frac{(6n+1)\pi}{3} \right)$$
  
 Where  $n$  is integer

q) Separate into real & imaginary parts the expression  $i^i$ .

→ Sol<sup>n</sup>:  $i^i = e^{\log \{i^i\}}$

$$= e^{i^i \log i} \quad \text{--- (1)}$$

$$i^i = e^{\log i^i}$$

$$= e^{i \log i}$$

$$= e^{i \left\{ \log 1 + i \left( 2m\pi + \frac{\pi}{2} \right) \right\}}$$

$$= e^{i \left\{ i \left( 2m\pi + \frac{\pi}{2} \right) \right\}}$$

$$= e^{- \left( 2m\pi + \frac{\pi}{2} \right)}$$

Let  $\log i = \left\{ \log 1 + i \left( 2m\pi + \frac{\pi}{2} \right) \right\}$   
 $= i \left( 2m\pi + \frac{\pi}{2} \right)$

Substituting for  $i^i$  and  $\log i$  in (1), we get

$$i^i = e^{- \left( 2m\pi + \frac{\pi}{2} \right) \cdot i \left( 2m\pi + \frac{\pi}{2} \right)}$$

$$= e^{i\theta}$$

$$= \cos \theta + i \sin \theta$$

$$\left. \begin{array}{l} \text{where } \theta = \\ \left( 2m\pi + \frac{\pi}{2} \right) \end{array} \right\} e^{- \left( 2m\pi + \frac{\pi}{2} \right)}$$



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